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Cosmic String in the NUT–Kerr–Newman Spacetime

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We study the equilibrium configurations of a cosmic string described by the Nambu action in the NUT–Kerr–Newman spacetime which includes as special cases the Kerr–Newman black hole spacetime as well as the NUT spacetime which is considered as a cosmological model. In this study it is interesting to note that one can obtain parallel results for the Kerr–Newman black hole as well as the NUT spacetime.

1. INTRODUCTION

Cosmic strings are topologically stable objects which might be formed during a phase transition in the early universe (Kibble, 1976). The cosmic strings formed during a phase transition in the early universe might provide the seeds needed for galaxy formation (Zeldovich, 1980). Such strings are predicted in certain grand unified theories.

Recently Frolov *et al.* (1989) studied the possible equilibrium configurations of a cosmic string in curved spacetime such as the Kerr–Newman black hole spacetime. In this paper we study the equilibrium configurations of a cosmic string in the NUT–Kerr–Newman spacetime which includes as special cases Kerr–Newman black hole spacetime as well as NUT spacetime. The NUT spacetime has very interesting properties.

2. MOTION OF A STRING

In the approximation that the gravitational field of the string is neglected, the motion of the string is described by the Nambu action (Vilenkin, 1985; Nambu, 1969, 1970; Neilsen and Olesen, 1973; Maeda and Turok, 1987)

$$I = -\mu \int d^2 l \left[-\det\left(g_{\alpha\beta} \frac{\partial x^\alpha}{\partial l^a} \frac{\partial x^\beta}{\partial l^b} \right) \right]^{1/2} \tag{1}$$

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where μ is the mass of the string per unit length, $g_{\alpha\beta}$ ($\alpha, \beta = 1, 2, 3, 4$) is an external gravitational field, and l^a denotes the world-sheet coordinates ($a, b = 0, 1$; $l^0 = \tau$, $l' = \sigma$). We consider the string to be open and infinite. In this case we also suppose that the force is applied to the string at infinity so that the string will not fall to the source responsible for creating the spacetime concerned.

In general a stationary spacetime is given by

$$ds^2 = -R(dt^2 + L_i\, dx^i)^2 + \frac{1}{R}\, l_{ij}\, dx^i\, dx^j \tag{2}$$

where $\partial_t R = \partial_t L_i = \partial_t l_{ij} = 0$ and $i, j = 2, 3, 4$. For time-independent string configurations where $\tau = t$ and the spacelike coordinates x^i depend on σ, the Nambu action can be written as

$$I = -\mu \int d\sigma \left(l_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} \right)^{1/2} \Delta t \tag{3}$$

Since the equilibrium configurations correspond to minimal energy, the problem is reduced to the investigation of the geodesics in a three-dimensional space with the metric

$$ds^2 = l_{ij}\, dx^i\, dx^j \tag{4}$$

3. EQUILIBRIUM CONFIGURATION

For the NUT–Kerr–Newman geometry we have

$$R = \frac{\Delta - a^2 \sin^2 \theta}{r^2 + (n - a\cos\theta)^2} \tag{5}$$

$$L = \delta_i^\mu L_\mu \tag{6}$$

where

$$L_\mu = \frac{a \sin^2 \theta (2Mr - e^2 + n^2) + \Delta(n^2/a - 2n \cos\theta)}{\Delta - a^2 \sin^2 \theta}$$

$$\Delta = r^2 - 2Mr + a^2 + e^2 - n^2$$

and M, a, e, and n are the mass, angular momentum per unit mass, charge, and NUT (magnetic mass) parameters, respectively. The three-dimensional metric l_{ij} is given by

$$l_{ij}=0 \qquad \text{where} \quad i\neq j \tag{7a}$$

$$l_{rr}=\frac{\Delta-a^2\sin^2\theta}{\Delta} \tag{7b}$$

$$l_{\theta\theta}=\Delta-a^2\sin^2\theta \tag{7c}$$

$$l_{\phi\phi}=\Delta\sin^2\theta \tag{7d}$$

For our study of the geodesics of the three-dimensional space metric l_{ij} we will use the Hamilton–Jacobi method (Misner *et al.*, 1973). We can write the Hamilton–Jacobi equation of the metric l_{ij} as

$$\frac{\partial S}{\partial\sigma}+\frac{1}{2}l^{ij}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j}=0 \tag{8}$$

where σ is an affine parameter along the geodesic. If we write

$$S=-\tfrac{1}{2}q^2\sigma+k\phi+P(r)+Q(\theta) \tag{9}$$

then we have from (7)–(9)

$$\Delta\left(\frac{dP}{dr}\right)^2-\frac{a^2k^2}{\Delta}-q^2\Delta=-m^2 \tag{10a}$$

$$\left(\frac{dQ}{d\theta}\right)^2+\frac{k^2}{\sin^2\theta}+q^2a^2\sin^2\theta=m^2 \tag{10b}$$

where m^2 is the separation constant. The integral of motion k corresponds to the Killing vector $\eta_\phi=\partial/\partial\phi$ and m is related to the existence of the Killing tensor η_{ij}:

$$m^2=\eta^{ij}p_ip_j \tag{11}$$

where

$$p_i=\frac{\partial S}{\partial x^i} \tag{12}$$

and

$$\eta^i_j=\text{diag}(a^2\sin^2\theta,\Delta,\Delta+a^2\sin^2\theta) \tag{13}$$

On integration from (10) we have

$$P(r)=\int^{r} dr\sqrt{H} \tag{14a}$$

$$Q(\theta)=\int^{\theta} d\theta\sqrt{\Theta} \tag{14b}$$

where

$$H=\frac{a^2k^2}{\Delta^2}-\frac{m^2}{\Delta}+q^2 \tag{15}$$

and

$$\Theta=m^2-\frac{k^2}{\sin^2\theta}-q^2a^2\sin^2\theta \tag{16}$$

Therefore equation (9) can be written as

$$S=\tfrac{1}{2}q^2\sigma+k\phi+\int^{r}\sqrt{H}\,dr+\int^{\theta}\sqrt{\Theta}\,d\theta \tag{17}$$

By differentiating (17) with respect to q^2, m, and k and setting each of the derivatives equal to zero, we obtain the equations

$$\sigma-\sigma_0=\int_{r_0}^{r}\frac{dr}{\sqrt{H}}-a^2\int_{\theta_0}^{\theta}\frac{\sin^2\theta}{\sqrt{\Theta}}\,d\theta \tag{18}$$

$$\int_{r_0}^{r}\frac{dr}{\Delta\sqrt{H}}=\int_{\theta_0}^{\theta}\frac{d\theta}{\sqrt{\Theta}} \tag{19}$$

$$\phi-\phi_0=k\left(\int_{\theta_0}^{\theta}\frac{d\theta}{\sin^2\theta\sqrt{\Theta}}-a^2\int_{r_0}^{r}\frac{dr}{\Delta^2\sqrt{H}}\right) \tag{20}$$

Equations (18)–(20) describe the equilibrium configuration of a string passing through the point (r_0, θ_0, ϕ_0) where the value of the affine parameter σ is σ_0. The string lies on a rotational surface given by equation (19). Equation (20) provides a unique curve on this surface. Since q is an inessential parameter, it can be changed by redefinition of the affine parameter σ. From now on we set $q=1$.

Equations (18)–(20) can be put in the following form:

$$p_r^2 = \left(l_{rr} \frac{dr}{d\sigma}\right)^2 = H \tag{21}$$

$$p_\theta^2 = \left(l_{\theta\theta} \frac{d\theta}{d\sigma}\right)^2 = \theta \tag{22}$$

$$p_\phi^2 = \left(l_{\phi\phi} \frac{d\phi}{d\sigma}\right)^2 = k^2 \tag{23}$$

To analyze the form of the rotational surface we rewrite (22) as

$$p_\theta^2 = m^2 - \frac{k^2}{\sin^2\theta} - a^2 \sin^2\theta = m^2 - V(\theta)$$

where

$$V(\theta) = \frac{k^2}{\sin^2\theta} + a^2 \sin^2\theta$$

If $m^2 = V(\theta)$ is the minimal value of the function $V(\theta)$ at $\theta = \theta_0$, the solution of (22) is $\theta = \theta_0$. In this particular case the surface on which the string lies is conelike.

When $k < a$, we get $\theta_0 = \arcsin(|k/a|^{1/2})$ and $V(\theta_0) = 2a|k|$. In the case $k > a$ we get $\theta_0 = \pi/2$ and $V(\theta_0) = k^2 + a^2$. For $k^2 > a^2$ and $m^2 = k^2 + a^2$, the string lies in the equatorial plane of the source responsible for the NUT–Kerr–Newman spacetime.

4. DISCUSSION

The results obtained in this paper apply for the NUT spacetime for $a = e = 0$ and for the Kerr–Newman geometry for $n = 0$.

This study not only encompasses the known results of Frolov *et al.* (1989) in the context of the Kerr–Newman black hole, but also provides similar results for the NUT spacetime, which is considered as a homogeneous anisotropic cosmological model (Misner *et al.*, 1973). This work can also be extended to the NUT–Kerr–Newman–de Sitter spacetime, which may be interesting from the point of view of the inflationary scenario of the early universe.

REFERENCES

Frolov, V. P., Skarzhinsky, V. D., and Zelnikov, A. I. (1989). *Physics Letters B*, **224**, 255.
Kibble, T. W. B. (1976). *Journal of Physics A*, **9**, 1387.

Maeda, K., and Turok, N. (1987). Fermilab Preprint 87/209-A.
Misner, C., Thorne, K., and Wheeler, J. A. (1973). *Gravitation*, Freeman, San Francisco.
Misner, C. W. (1963). *Journal of Mathematical Physics*, **4**, 924.
Nambu, Y. (1969). *Proceedings International Conference on Symmetrics and Quark Models*, Wayne State University.
Nambu, Y. (1970). Lectures at Copenhagen Symposium.
Neilsen, H., and Olesen, P. (1973). *Nuclear Physics B*, **61**, 45.
Vilenkin, A. (1985). *Physics Reports*, **121**, 263.
Zeldovich, Ya. B. (1980). *Monthly Notices of the Royal Astronomical Society*, **192**, 663.